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A Discrete Hamilton-Pontryagin approach to the statics of Kirchhoff Rods

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Abstract

In this work we address the problem of computing stable static equilibria of Kirchhoff rods under different boundary conditions and possibly subject to contact constraints. Our approach relies on formulating the continuous problem as an Optimal Control Problem (OCP) and discretizing it using direct methods of numerical optimal control. Conceptually our approach is similar to the one developed in [3] in that we also leverage constructions from the discrete mechanics of rigid bodies ([1] in our case) in conceiving numerical schemes for the statics of rods.

A Kirchhoff rod is a thin elastic rod characterized by small strains (linear elasticity), large displacements and finite rotations (geometrical non linearities). It is assumed perfectly inextensible, and undergoes only pure bending and twisting deformations. The centerline of a Kirchhoff rod of length L is parametrized by a curve r mapping the arclength parameter $s \in [0,L]$ to $\mathbb{R}^3$. The orientation of the cross sections is given by a material frame varying along the curve r and represented here as a rotation matrix $R(s) \in SO(3)$. As in [2] the configuration space of our geometrically exact rod is the special Euclidean group SE(3). Accordingly the state of our OCP is $(R,r) : [0,L] \to SE(3)$. We collect the bending strains κ_1, κ_2, and the twisting strain τ in a vector $\kappa := (\kappa_1, \kappa_2, \tau)^T$ which will be interpreted as the control input in our OCP. The kinematics of the material frame read $R' = R\hat{\kappa}$, where $\hat{\kappa}$ is the skew symmetric matrix $\hat{\kappa} = \begin{bmatrix} 0 & -\tau & \kappa_2 \\ \tau & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix}$ and prime(') denotes derivation with respect to the arclength s .

The Euler-Bernoulli constraint reads $r' = Re_3$, where $e_3 = (0,0,1)^T$; it encodes the incompressibility and no shear conditions, and couples the frame to the centerline (the frame is said to be adapted to the centerline). The SE(3) structure of the problem becomes apparent by rewriting the frame kinematics and Euler-Bernoulli constraint as the SE(3) reconstruction equation,

$$\frac{d}{ds}\begin{bmatrix} R(s) & r(s) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(s) & r(s) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\kappa} & e_3 \\ 0 & 0 \end{bmatrix}. \tag{1}$$

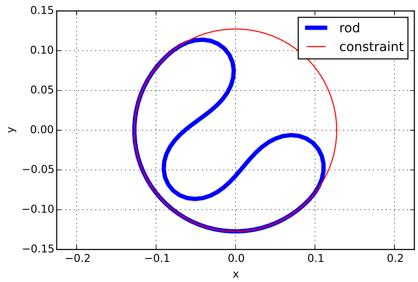
Finding the stable static equilibria of a Kirchhoff rod subject to boundary conditions and contact constraints can then be formulated as the OCP

$$\begin{aligned} \min_{R,r,\kappa} \quad & \int_0^L \mathscr{L}(R(s), r(s), \kappa(s))\mathrm{d}s \\ \text{s. t.} \quad & R' = R\hat{\kappa} \\ & r' = Re_3 \\ & g_I(R,r,\kappa) \leq 0 \\ & g_{bd}(R(0), r(0), R(L), r(L)) = 0, \end{aligned} \tag{2}$$

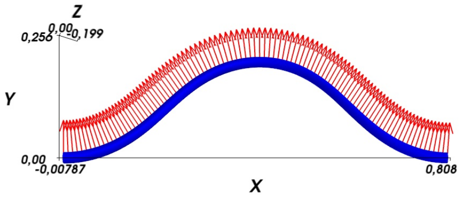
where the 'Lagrange cost' models in our example the sum of bending twisting and gravitational potential energies $\mathscr{L} = \frac{1}{2}(EI_1\kappa_1(s)^2 + EI_2\kappa_2(s)^2 + \mu J\tau(s)^2) + U_g(r(s))$, while g_I models inequality constraints and g_{bd} encodes the boundary conditions. Note that finding stable static equilibria of Cosserat rods could be formulated similarly, it would amount formally to doing a regularization of the Euler Bernoulli constraint: one would just have to add a shear and stretch degree of freedom $v : [0,L] \to \mathbb{R}^3$, an extra quadratic term $\frac{1}{2}(v - e_3)^T C(v - e_3)$ in the elastic energy and to modify the kinematics as $r' = Rv$.

From the point of view of numerical optimal control there are two major categories of ways to find local solutions of the problem (2), namely *indirect methods* and *direct methods*. The indirect approach would be to first write the (infinite dimensional) first order optimality conditions of the OCP (2) and then to discretize it. In the absence of inequality constraints one would retrieve the Kirchhoff balance equations. However in the presence of inequality constraints the first order optimality conditions take on the form of a non smooth boundary value problem which is difficult to solve and one would not be guaranteed to find local minima but rather saddle points. In our opinion it is simpler to tackle the problem using direct methods, where one first discretizes the OCP (2), turning it into a finite dimensional Non Linear Program (NLP), and then optimizing it using standard NLP software like IPOPT [4].

From the point of view of direct methods of numerical optimal control the easiest technique to try is 'direct single shooting', where the controls (ie the strains κ) are discretized into a suitable finite dimensional space and the states (R,r) are retrieved as functions of the strains by numerically solving the reconstruction equation in the OCP (2). In this case one then retrieves a 'strain based' finite element approach. Using piecewise constant strains and a Lie-Euler integrator on $SE(3)$ for the kinematics, the formulation coincides precisely with the Super-Helix element for



(a)



(b)

Figure 1: (a) Rod equilibrium under periodic boundary conditions and inequality constraints. (b) Rod equilibrium under fixed-fixed boundary conditions.

the statics of Kirchhoff rods [6]. More interestingly, the optimal control framework suggests a way to tackle higher order strains by simply using higher order Lie group integrators, for example general Runge Kutta Munthe-Kaas methods as employed in [1] in the similar context of Hamilton Pontryagin mechanics on Lie groups. The second branch of direct methods is 'direct multiple shooting' where both the strains κ and the states (R,r) are discretized, leading to a *mixed formulation*. The NLP in the mixed formulation is of higher dimensionality but with more sparsity and simpler non linearities. Using Lie group methods to discretize the kinematics allows us to avoid formulating supplementary orthogonality constraints for the frame R_i at each node s_i. In Figure (1b) we obtain a stable static equilibrium of a Kirchhoff rod subject to fixed-fixed boundary conditions, the director vectors are shown as red arrows. In Figure (1a) we obtain a stable static equilibrium configuration for a planar Kirchhoff rod with periodic boundary conditions and subject to the inequality constraint of remaining inside a circle, this curve can be found for example experimentally as the profile of a paper cylinder packed into a smaller cylinder of radius ρ_c. Using the mixed formulation allows to keep the inequality constraints $||r_i||^2 \leq \rho_c^2$ convex albeit nonlinear. The simulation used 100 elements, total simulation time was 1.3s on a 2.6GHz Intel Core i7 processor parting from a random initial configuration using IPOPT. Our proposed method is quite robust to the choice of the initial configuration and could thus be used to initialize load displacements analyses. In conclusion the Optimal Control point of view provides a very useful theoretical and numerical framework to conceive, analyse and implement strain based and mixed finite element discretizations of rod statics under constraints.

Acknowledgments

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